Ricci flow, Wallach Spaces and High school calculus

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- Why do we care about Ricci flow?
- Ricci flow is a basic tool to solve the Poincare Conjecture.

- What is Poincare Conjecture?

Poincare Conjecture

- Conjectured by Henri Poincare in 1904
- Is one of the **Millennium Prize Problems**

  i.e. can get US $1,000,000 if you solved it!
- A proof of this conjecture was given by Grigori Perelman in 2003; its review was completed in August 2006.
- What is the statement?

Every simply connected, closed 3-manifold is homeomorphic to the 3-sphere.

Poincare Conjecture

Other dimensions

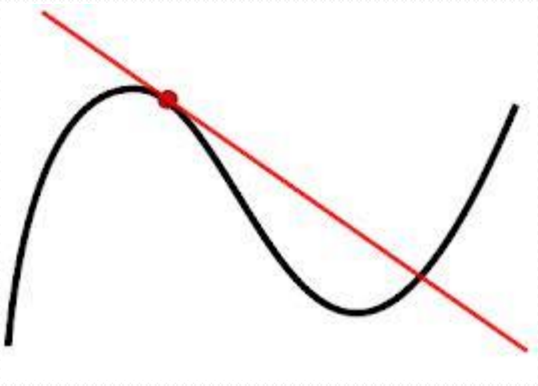
- dim = 1,2, known for a long time
- In 1961, Stephen proved the Generalized Poincaré conjecture for dimensions > 4.
- In 1982 Michael Freedman proved the Poincaré conjecture in dimension four.
- So only original dimension 3 case left since then.

A few terms to go over

- Tangent
- Curvature
- Division algebra

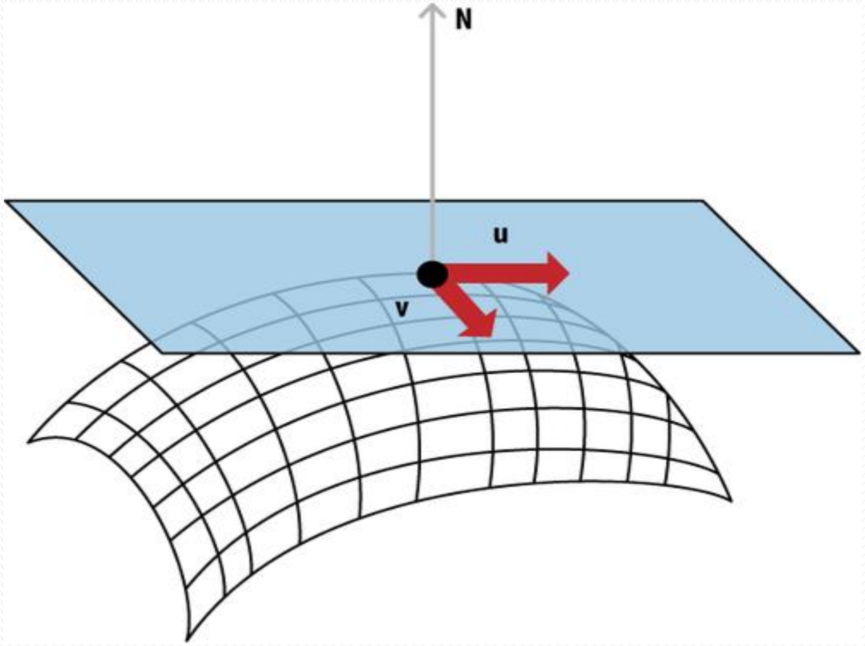
Tangent

- Tangent line (for curves)



At each point, get a line
1 dimension!

- Tangent plane (for surfaces)

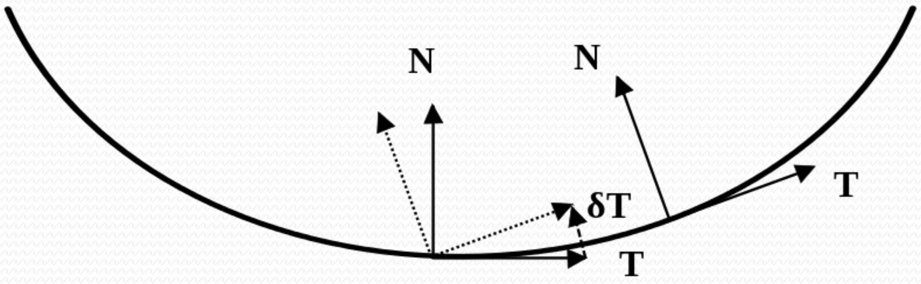


- 2 dimension!
- Vectors u, v
- Imagine it is $\mathbb{R}^2$

- For space of higher dimension (dim n), we have tangent space of dimension n.
- Can Imagine we have $\mathbb{R}^n$ attaching to each point

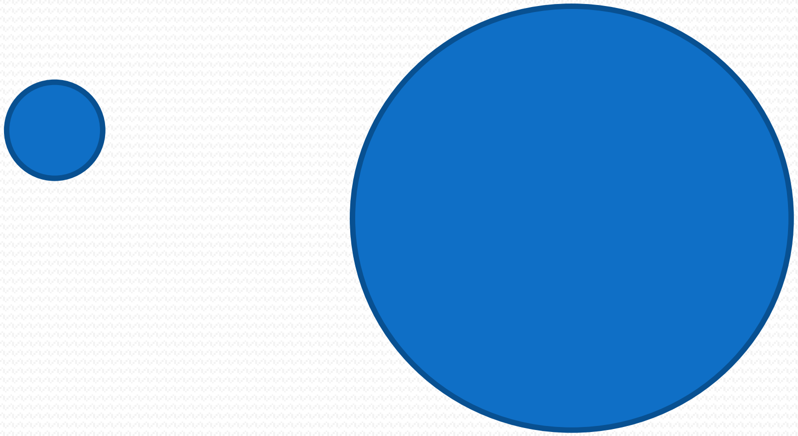
Curvature

- Curvature of a plane curve
  - Tells you how 'curvy' a curve is
- How much it bends away from tangent



- Curvature of straight line
  - Not curvy. So the curvature = 0.

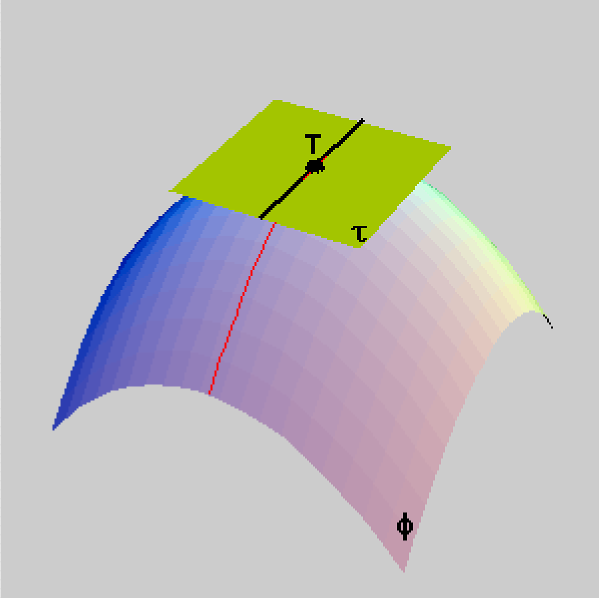
- Curvature of a circle
  - A circle with larger radius. It seems to be more flat
  - So for a circle of radius r, curvature $= \frac{1}{r}$

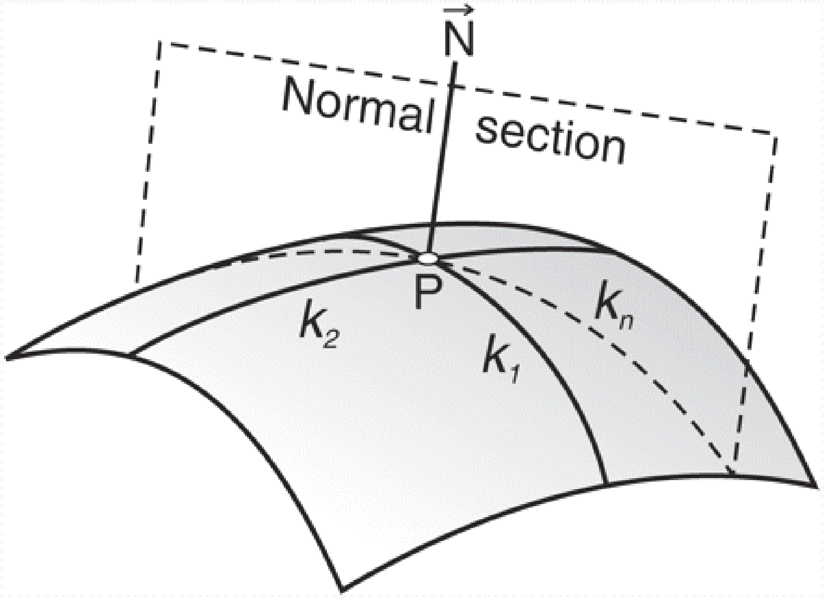


Note: Every point on the circle have the same curvature. Conversely, if a curve without boundary is of constant curvature, it is a circle.

Curvature of a surface

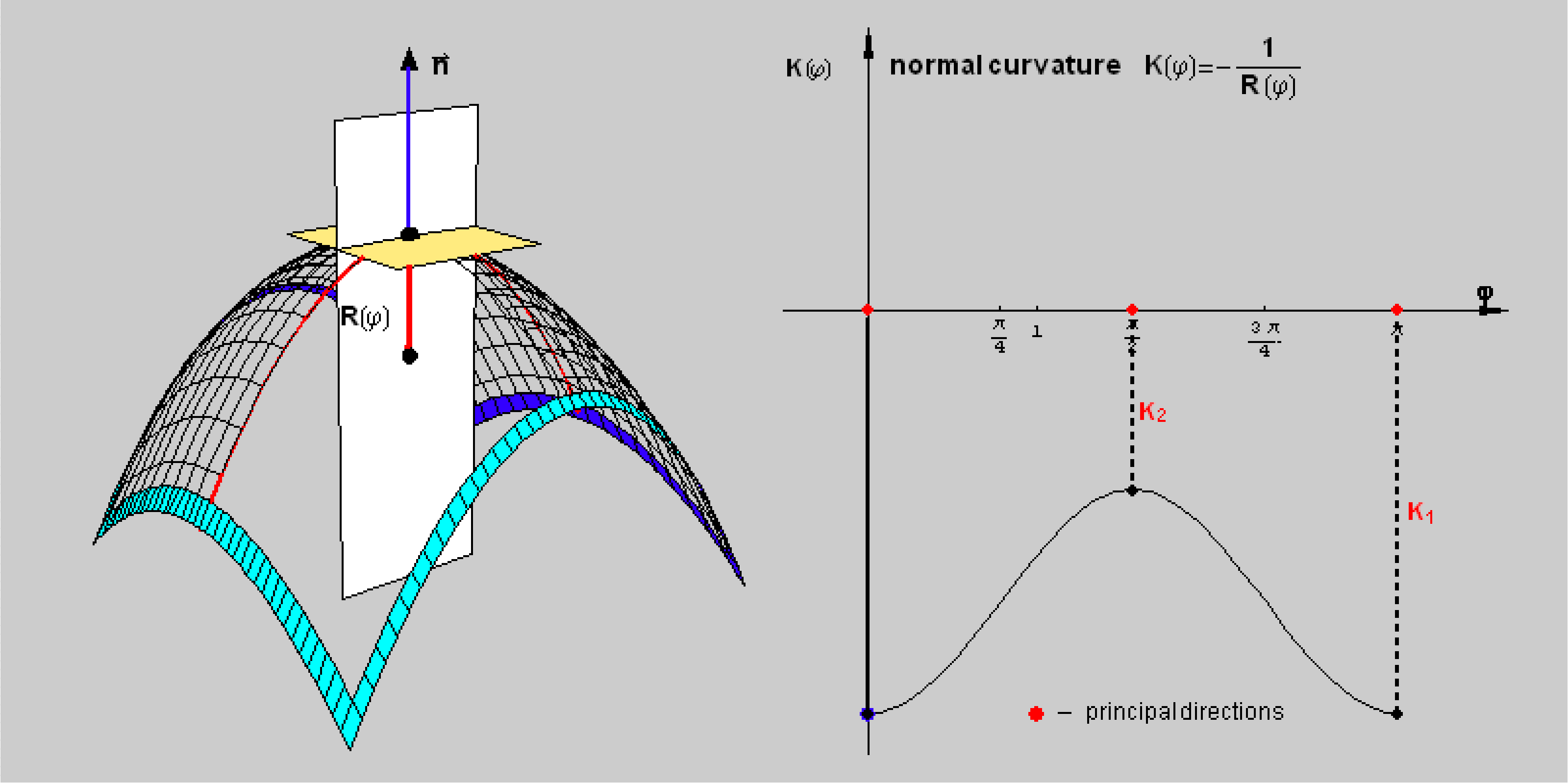
- Want to generalize what we have for curve





- Cut the surface by the plane generated by normal vector and another vector. Get a curve

Normal curvature



Gaussian curvature

- Take the max and the min normal curvature
  - Gaussian curvature = max × min.
- Gaussian curvature of a plane = 0
- A sphere has constant positive curvature.
- Again if a simply connected ‘nice’ surface is of constant positive curvature, it ‘is’ a sphere.

Higher Dimension

- Higher dimension, we generalize Gaussian curvature to sectional curvature.
- Take a 2-plane ($\mathbb{R}^2$) in the tangent space ($\mathbb{R}^n$), calculate the 'Gaussian curvature' associated to the 2-plane
- Again if a simply connected 'nice' surface is of constant positive curvature, it 'is' a sphere.

- Ricci Curvature
  - Measure the local deformation of an n-sphere.

Real division algebras

- Real number $\mathbb{R}$: can do $+, -, \times, -$
- Ask: can we extend this?
- Get complex number $\mathbb{C}$
  - $a + bi$, where a, b are real numbers.
  - Can think of it as $\mathbb{R}^2$
  - Conjugate $\overline{a + bi} = \mathrm{a} - \mathrm{bi}$
- Further extend, we have Quaternions $\mathbb{H}$
  - $a + bi + cj + dk$, can think of as $\mathbb{R}^4$
- One more: Octonion $\mathbb{O}$
  - Similarly, can think of as $\mathbb{R}^8$
- That's all!

Ricci Flow

- Introduce by Hamilton Ricci.
- A procedure for transforming irregular spaces into uniform ones.
- Over time, the irregularly curved dumbbell relaxes into a uniformly curved surface, in a 2-dimensional version of the Ricci flow.
- Is basically a differential equation. We will come back to the equation later.

How is Poincare Conjecture related to Ricci flow?

- Hamilton proved that a compact 3-manifold with positive Ricci curvature is deformed to a space of constant positive sectional curvature.
- This implies that if a simply connected compact 3-manifold has a metric with positive Ricci curvature, it is diffeomorphic to the sphere $\mathbb{S}^3$.

Goal!

- Under Ricci flow, the Wallach spaces

1. With strictly positive sectional curvature turns into mixed sectional curvature
2. With positive Ricci tensor would turns to negative
3. Not all of them would turn from positive sectional curvature to negative Ricci

What are Wallach Varieties?

- Consider the spaces M=G/K , where

G	K
$SU(3)$	T^2
$Sp(3)$	$Sp(1) \times Sp(1) \times Sp(1)$
F_4	$Spin(8)$

- Equivalently, with $M = G/K$, we can think as

G	K
$U(3, \mathbb{C})$	$U(1, \mathbb{C}) \times U(1, \mathbb{C}) \times U(1, \mathbb{C})$
$U(3, \mathbb{H})$	$U(1, \mathbb{H}) \times U(1, \mathbb{H}) \times U(1, \mathbb{H})$
$F_4" = "U(3, \mathbb{O})$	$Spin(8)" = "U(1, \mathbb{O}) \times U(1, \mathbb{O}) \times U(1, \mathbb{O})$

- The tangent space

$$T(M)_{eK} = \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F} = \mathbb{R}^d \oplus \mathbb{R}^d \oplus \mathbb{R}^d,$$ where

$\mathbb{F} = \mathbb{C}, \mathbb{H}, \mathbb{O}$, and $d = 2,4,8$ respectively

Space	Tangent space	Dimension
1	$\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$ $(\mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R}^2)$	6
2	$\mathbb{H} \oplus \mathbb{H} \oplus \mathbb{H}$ $(\mathbb{R}^4 \oplus \mathbb{R}^4 \oplus \mathbb{R}^4)$	12
3	$\mathbb{O} \oplus \mathbb{O} \oplus \mathbb{O}$ $(\mathbb{R}^8 \oplus \mathbb{R}^8 \oplus \mathbb{R}^8)$	24

Set Up

- Metric ('inner ' product on the tangent space $\mathbb{R}^n$)
  - $\mathbb{R}^n$: $< v_1, v_2 > = x_1 y_1 + x_2 y_2 + \cdots$
- $\langle \dots, \dots \rangle_{eK} = x_1 \langle \dots, \dots \rangle_1 + x_2 \langle \dots, \dots \rangle_2 + x_3 \langle \dots, \dots \rangle_3$,
  where $x_i > 0$ and $\langle z, w \rangle_i = Re\, z\, \overline{w}$.
  Represent it as (x_1, x_2, x_3).
- (Wallach, 1972)
  - If $x_1 = x_2$,
  - then the sectional curvature is strictly positive if $0 < \frac{x_3}{x_1} < 1$ or $1 < \frac{x_3}{x_1} < \frac{4}{3}$ and there is some strictly negative curvature if $\frac{x_3}{x_1} > \frac{4}{3}$.

- At that time, no compact homogeneous spaces with strictly positive sectional curvature are known.
- Wallach has given this 3 new examples (at that time) of compact even dimensional homogeneous spaces admitting homogeneous Riemannian structures of strictly positive curvature.
- Until now, the 6 dim space is still the known space with lowest dimension and the 24 dim is the example of highest dimension

- Corresponding to (x_1, x_2, x_3), the Ricci curvature

$$r_1 x_1 \langle \ldots, \ldots \rangle_1 + r_2 x_2 \langle \ldots, \ldots \rangle_2 + r_3 x_3 \langle \ldots, \ldots \rangle_3$$

- Where r_i are calculated as

$$r_i = \frac{d\, x_i^2 \ - d\, x_j^2 - d\, x_k^2 + (10d - 8)x_j x_k}{2x_1 x_2 x_3}$$

where $d = 2,4,8$ and $\{i, j, k\} = \{1,2,3\}$.

Ricci flow equation

- In this setting, the Ricci flow eqn. can been given as

$$\frac{dx_i}{dt} = -2\, r_i x_i$$

- The goal is to say what happens to positive sectional curvature or Ricci curvature under the above non-linear ODE.
- Note the set of metrics with $x_i = x_j$ are preserved by the Ricci flow.
- Permutation of the indices of x_i also preserves the solutions

Sectional curvature

- Start with a metric with $x_1 = x_2 = 1$.
- Recall there is some negative sectional curvature if the metric $(x_1, x_2, x_3) = (1,1,u)$ with $u > \frac{4}{3}$
- i.e. want $u = \frac{x_3}{x_1}$ increase from $\frac{4}{3}$
- Question: Want $\frac{x_3}{x_1}$ increase means want
$$\frac{d}{dt}\bigg|_{t=0} \frac{x_3(t)}{x_1(t)} > 0, = 0, < 0?$$
- > 0

- To prove some curvature turns negative, it suffices to start from initial metric as $\left(1, 1, \frac{4}{3}\right)$ and check

$$\frac{d}{dt}\Big|_{t=0} \frac{x_3(t)}{x_1(t)} > 0$$

- How to $\frac{d}{dt}\frac{x_3(t)}{x_1(t)}$?
- Quotient rule!

$$\frac{d}{dt}\frac{x_3(t)}{x_1(t)} = \frac{x_3'(t)\, x_1(t) - x_3(t)\, x_1'(t)}{x_1(t)^2}$$

$$\frac{d}{dt}\frac{x_3(t)}{x_1(t)} = -2\,\frac{x_3(t)}{x_1(t)}(r_3 - r_1)$$

$$\frac{dx_i}{dt} = -2\, r_i x_i$$

- $-2(r_3 - r_1)|_{t=0} = -2 + \frac{4d}{3} > 0$
- Theorem (-, Wallach)

On the Wallach spaces, the Ricci flow deforms certain positively curved metrics into metrics with mixed sectional curvatures.

- Further look in the case $x_1 = x_2$
- $2(r_1 - r_3) = \frac{-2\left(1-\frac{x_3}{x_1}\right)\left(4d-4-d\,\frac{x_3}{x_1}\right)}{x_1 x_3}$
- Look at the critical point (i.e. $\frac{d}{dt}\frac{x_3(t)}{x_1(t)} = 0$)

- This implies that if $1 < \frac{x_3(t)}{x_1(t)} < \frac{4(d-1)}{d}$, then

$$\lim_{t\to\infty} \frac{x_3(t)}{x_1(t)} = \frac{4(d-1)}{d} \text{ under the Ricci flow}$$

- As $\frac{4}{3} < \frac{4(d-1)}{d}$ for $d = 2,4,8$, we still get the above theorem
- Note the line for $\frac{x_3}{x_1}$ goes from 1 to $\frac{4(d-1)}{d}$ gives a full set of Einstein metrics for $x_1 = x_2$. Hence this tell us the Ricci flow flows from one Einstein metric to another one

Thank you!

Imaginary end